RECONSTRUCTION OF THE THERMAL CONDUCTIVITY COEFFICIENT

FROM THE SOLUTION OF THE NONLINEAR INVERSE PROBLEM

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We consider the iteration algorithm for the solution of the nonlinear inverse problem of non-steady-state heat conduction.

In the analysis of data on the non-steady-state thermophysical experiments, one has to solve the nonlinear inverse heat-conduction problems for the coefficients. In this case, one needs to determine the temperature dependence of one or several coefficients in the heat equation from the results of temperature measurements at several points of the body in question [1].

Below we consider the nonlinear inverse problem for the coefficients in which we reconstruct the temperature dependence of the thermal conductivity coefficient from the temperature data at one internal point of an infinite planar film. At the edge of the film, boundary conditions of the first kind are known.

The inverse problem is formulated as follows. It is required to determine the functions $T(x, \tau)$ and $\lambda(T)$ from the conditions

$$C(T)\frac{\partial T}{\partial \tau}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right),\ 0<x<b,\ 0<\tau\leqslant\tau_m, \tag{1}$$

$$T(x,\ 0)=T_0,\ 0\leqslant x\leqslant b, \tag{2}$$

$$T(0,\ \tau)=\varphi_1(\tau), \tag{3}$$

$$T(b,\ \tau)=\varphi_2(\tau), \tag{4}$$

$$T(d,\ \tau)=f(\tau),\ 0<d<b, \tag{5}$$

where $C(T)$, $\varphi_1(\tau)$, $\varphi_2(\tau)$, and $f(\tau)$ are known functions. The solution of the inverse problem (1)-(5) will be sought from the condition of the least rms deviation

$$J=\int\limits_0^{\tau_m}[T(d,\ \tau,\ \lambda(T))-f(\tau)]^2\,d\tau. \tag{6}$$

Since Eq. (1) and the given boundary conditions of the first kind are homogeneous, the region of definition of the required function $\lambda(T)$ is known. We represent this region in the form of an interval $D = [T_{min}, T_{max}]$. The interval D will be divided into m equal parts, and we introduce the net

$$w=\{T_k=T_{\min}+k\Delta T,\ k=-2,\ -1,\ \ldots,\ m+3;$$
$$\Delta T=(T_{\max}-T_{\min})/m\}.$$

The unknown function $\lambda(T)$ will be approximated by a B spline on the net w [2]:

$$\lambda(T)=\sum_{k=-1}^{m+1}\lambda_k B_k(T), \tag{7}$$

where

$$B_k(T)=B_0(T-k\Delta T);$$

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$$B_0(T)=\frac{1}{\Delta T^3}[(T+2\Delta T)_+^3-4(T+\Delta T)_+^3+6(T)_+^3-4(T-\Delta T)_+^3+(T-2\Delta T)_+^3];$$

$$(T-\Delta T)_+^3=[\max(0,\ T-\Delta T)]^3.$$

Using representation (7), the inverse problem we have just formulated reduces to finding $(m + 3)$-dimensional vector consisting of the parameters $\lambda = \{\lambda_{-1},\ \lambda_0,\dots,\ \lambda_{m+1}\}$ from the condition of minimum functional (6), subject to restrictions (1)-(4).

The iteration process of minimization of the functional (6) will be constructed on the basis of the gradient methods. In particular, we use the method of conjugate gradients which is sufficiently economical in the number of numerical calculations, and is very effective for a wide class of response functions [3].

We obtain a formula for the calculation of the gradient of the functional (6). The boundary-value problem (1)-(4) will be represented as a heat-conduction problem for a two-layer infinite film with identical thermophysical properties of the layers, and an ideal contact between them. Using (7) we then find

$$C(T)\frac{\partial T_i}{\partial\tau}=\frac{\partial}{\partial x}\left[\left(\sum_{k=-1}^{m+1}\lambda_k B_k(T)\right)\frac{\partial T_i}{\partial x}\right],$$

$$X_{i-1}<x<X_i,\ \ 0<\tau\leqslant\tau_m;\ i=1,\ 2;\ X_0=0<X_1=d<X_2=b, \tag{8}$$

$$T_i(x,\ 0)=T_0,\ X_{i-1}\leqslant x\leqslant X_i, \tag{9}$$

$$T_1(0,\ \tau)=\varphi_1(\tau), \tag{10}$$

$$T_1(d,\ \tau)=T_2(d,\ \tau), \tag{11}$$

$$\frac{\partial T_1(d,\ \tau)}{\partial x}=\frac{\partial T_2(d,\ \tau)}{\partial x}, \tag{12}$$

$$T_2(b,\ \tau)=\varphi_2(\tau). \tag{13}$$

Following [4] we will assume that the components of the vector $\bar{\lambda}$ acquire small increments $\Delta\lambda_k$, $k = 1, 0,\dots, m + 1$. The temperature $T(x, \tau)$ is then increased by $\vartheta(x, \tau)$. It can be shown that in the linear approximation, the function $\vartheta(x, \tau)$ satisfies the following boundary-value problem:

$$C(T)\frac{\partial\vartheta_i}{\partial\tau}=\left(\sum_{k=-1}^{m+1}\lambda_k B_k(T)\right)\frac{\partial^2\vartheta_i}{\partial x^2}+2\frac{\partial T_i}{\partial x}\left(\sum_{k=-1}^{m+1}\lambda_k\frac{dB_k}{dT}\right)\frac{\partial\vartheta_i}{\partial x}+$$

$$+\left[\frac{\partial^2 T_i}{\partial x^2}\left(\sum_{k=-1}^{m+1}\lambda_k\frac{dB_k}{dT}\right)+\left(\frac{\partial T_i}{\partial x}\right)^2\left(\sum_{k=-1}^{m+1}\lambda_k\frac{d^2B_k}{dT^2}\right)-\right.$$

$$\left.-\frac{\partial T}{\partial\tau}\frac{dC}{dT}\right]\vartheta_i+\frac{\partial^2 T_i}{\partial x^2}\left(\sum_{k=-1}^{m+1}\Delta\lambda_k B_k(T)\right)+\left(\frac{\partial T_i}{\partial x}\right)^2\left(\sum_{k=-1}^{m+1}\Delta\lambda_k\frac{dB_k}{dT}\right),$$

$$X_{i-1}<x<X_i,\ 0<\tau\leqslant\tau_m,\ i=1,2, \tag{14}$$

$$\vartheta_i(x,\ 0)=0,\ X_{i-1}\leqslant x\leqslant X_i, \tag{15}$$

$$\vartheta_1(0,\ \tau)=0, \tag{16}$$

$$\vartheta_1(d,\ \tau)=\vartheta_2(d,\ \tau), \tag{17}$$

$$\frac{\partial\vartheta_1(d,\ \tau)}{\partial x}=\frac{\partial\vartheta_2(d,\ \tau)}{\partial x}, \tag{18}$$

$$\vartheta_2(b,\ \tau)=0. \tag{19}$$

The linear part of the increment of functional (6) has the form

$$\Delta J = 2\int_0^{\tau_m} [T(d,\ \tau,\ \bar{\lambda}) - f(\tau)]\,\vartheta(d,\ \tau)\,d\tau. \tag{20}$$

We consider the boundary-value problem conjugate to problem (8)-(13):

$$C(T)\frac{\partial\psi_i}{\partial\tau} = -\left(\sum_{k=-1}^{m+1}\lambda_k B_k(T)\right)\frac{\partial^2\psi_i}{\partial x^2},$$

$$X_{i-1} < x < X_i,\ 0 < \tau \leqslant \tau_m,\ i = 1,\ 2, \tag{21}$$

$$\psi_i(x,\ \tau_m) = 0,\ X_{i-1} \leqslant x \leqslant X_i, \tag{22}$$

$$\psi_1(0,\ \tau) = 0, \tag{23}$$

$$\psi_1(d,\ \tau) = \psi_2(d,\ \tau), \tag{24}$$

$$\left(\sum_{k=-1}^{m+1}\lambda_k B_k(T)\right)\left(\frac{\partial\psi_1(d,\ \tau)}{\partial x} - \frac{\partial\psi_2(d,\ \tau)}{\partial x}\right) = 2(T(d,\ \tau) - f(\tau)), \tag{25}$$

$$\psi_2(b,\ \tau) = 0. \tag{26}$$

The expression for the increment of functional (20) will be transformed as follows:

$$\Delta J = \int_0^{\tau_m}\left(\frac{\partial\psi_1(d,\ \tau)}{\partial x} - \frac{\partial\psi_2(d,\ \tau)}{\partial x}\right)\sum_{k=-1}^{m+1}\lambda_k B_k(T)\,d\tau,$$

or, using conditions (16), (17), and (24),

$$\Delta J = \int_0^d\int_0^{\tau_m}\frac{\partial}{\partial x}\left[\frac{\partial\psi_1}{\partial x}\,\vartheta_1\sum_{k=-1}^{m+1}\lambda_k B_k(T)\right]dx\,d\tau + \int_d^b\int_0^{\tau_m}\frac{\partial}{\partial x}\left[\frac{\partial\psi_2}{\partial x}\,\vartheta_2\sum_{k=-1}^{m+1}\lambda_k B_k(T)\right]dx\,d\tau. \tag{27}$$

Substituting relations (21), (14), (16)-(19), (23), and (24) into equality (27), we obtain, after some transformations,

$$\Delta J = \sum_{k=-1}^{m+1}\Delta\lambda_k\left\{\int_0^d\int_0^{\tau_m}\psi_1\left[\frac{\partial^2 T_1}{\partial x^2}B_k(T) + \left(\frac{\partial T_1}{\partial x}\right)^2\frac{dB_k}{dT}\right]dx\,d\tau + \int_d^b\int_0^{\tau_m}\psi_2\left[\frac{\partial^2 T_2}{\partial x^2}B_k(T) + \left(\frac{\partial T_2}{\partial x}\right)^2\frac{dB_k}{\partial T}\right]dx\,d\tau\right\}. \tag{28}$$

It follows from equality (28) that the formula for the components of the gradient vector of functional (6) has the form

$$J_k' = \int_0^d\int_0^{\tau_m}\psi_1\left[\frac{\partial^2 T_1}{\partial x^2}B_k(T) + \left(\frac{\partial T_1}{\partial x}\right)^2\frac{dB_k}{dT}\right]dx\,d\tau + \int_d^b\int_0^{\tau_m}\psi_2\left[\frac{\partial^2 T_2}{\partial x^2}B_k(T) + \left(\frac{\partial T_2}{\partial x}\right)^2\frac{dB_k}{dT}\right]dx\,d\tau,$$

$$k = -1,\ 0,\ \ldots,\ m+1. \tag{29}$$

If the gradient of the total functional is known, one can construct an iteration algorithm for the solution of the inverse problem for the coefficients. According to the method of conjugate gradients, the approximations are constructed from the formulas

$$\lambda_k^{p+1} = \lambda_k^p + \alpha_p g_k^p,\ k = -1,\ 0,\ \ldots,\ m+1,\ p = 0,\ 1,\ \ldots, \tag{30}$$

where

$$g_k^p = -J_k'^{(p)} + \beta_p g_k^{p-1};$$

$$\beta_0 = 0;\ \beta_p = \sum_{k=-1}^{m+1}(J_k'^{(p)} - J_k'^{(p-1)})J_k'^{(p)}\Big/\sum_{k=-1}^{m+1}(J_k'^{(p)})^2.$$

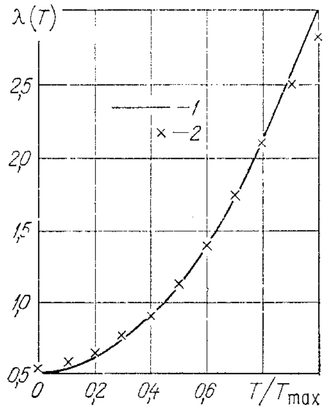

Fig. 1. Reconstruction of the temperature dependence of the thermal conductivity coefficient for "exact" input data. Curve 1 shows the exact values, and the points 2 are the reconstructed values (m = 3, p = 25).

The coefficient α_p determines the step at the p-th iteration, and can be calculated from the condition

$$\min_{\alpha} J(\bar{\lambda}^p + \alpha G^p).$$

In the present problem, the quantity α_p can be estimated explicitly. It can be shown (see, e.g., [5]) that the following relation holds:

$$\alpha_p = -\frac{\int_0^{\tau_m} [T(d, \tau, \bar{\lambda}^p) - f(\tau)]\, \vartheta(d, \tau)\, d\tau}{\int_0^{\tau_m} [\vartheta(d, \tau)]^2\, d\tau}. \tag{31}$$

The iteration process is now constructed as follows. The starting approximation of the required parameters is specified, and one solves problem (8)-(13). From the conjugate problem (21)-(26), the calculated approximate temperature field is used to calculate the gradient of the total functional from formulas (29). After solving the problem for increments (14)-(19), one determines the rate α_p, and relations (30) are used to find the new approximation. The process is subsequently repeated for the next approximation, etc. The process should be terminated according to the deviation [4], i.e., when the condition

$$J \leqslant \delta^2$$

holds, where $\delta^2 = \int_0^{\tau_m} \sigma_f^2(\tau)\, d\tau$ is the integral error of temperature specification at point x = d, and $\sigma_f(\tau)$ is the rms deviation of the input temperatures.

It should be noted that the above inverse heat-conduction problem is incorrectly formulated. The effect of the incorrectness can manifest itself in the instability of the numerical solution. To obtain stable results, one can use the principle of step regularization [1]. In this case, the regularity of the solution is achieved by decreasing the number of divisions of the temperature interval on which the required function λ(T) is defined.

The above algorithm for the reconstruction of the thermal conductivity coefficient from the non-steady-state temperature measurements was realized in the form of a computer program which we have used to calculate a number of methodical examples. The boundary-value problems (8)-(13), (14)-(19), and (21)-(26) were numerically solved using an implicit approximation scheme. In the process of solution of the problem (8)-(13), we used iterations with respect to the coefficients [6]. Some of the obtained results are shown in Fig. 1.

The boundary temperatures and the input data for the solution of the inverse problem were the temperatures obtained from the solution of the forward heat-conduction problem for boundary conditions of the second kind and a given thermal conductivity coefficient $\lambda(T) = 0.5 + 2T^2$. The remaining input data were chosen as follows: $q_1 = 1$, $q_2 = 0$, $C(T) = 1$, $b = 1$, $d = 0.5$, and $\tau_m = 1$.

In the spline approximation of the required function $\lambda(T)$ we made three divisions of the temperature interval $[T_{min}, T_{max}]$. The calculations were carried out on the difference net $n_x \times n_\tau = 20 \times 20$. For the 25 iterations we needed about 5 min of the processor time of the computer BÉSM-6. In this example, the problem was solved using exact data. The starting approximation for the heat-conduction coefficient was taken as a constant, and equal to $\lambda_0 = 0.75$. The obtained results demonstrate the sufficiently high efficiency of the suggested algorithm.

NOTATION

T, temperature; C(T), volume heat capacity; $\lambda(T)$, thermal conductivity coefficient; x, d, and X, coordinates; τ, time; b, right-hand boundary along x; τ_m, duration of the process; $f_i(\tau)$, input temperatures; $\vartheta(x, \tau)$, temperature increment; λ_k, k = 1, 0,..., m + 1, parameters in the spline approximation of the function $\lambda(T)$; B(T), B spline; α and β, parameters of the conjugate gradients method; J', gradient of the total functional; $\psi(x, \tau)$, conjugate variable; δ^2, integrated error of the input data; and p, number of iterations.

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CONDUCTIVITY OF MULTICOMPONENT HETEROGENEOUS SYSTEMS

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A method is proposed for calculating conductivity of a multicomponent heterogeneous system, taking account of its structure.

The conductivity Λ of a heterogeneous system is the coefficient in the linear relation between the average flux $\langle\vec{j}\rangle$ and the average value of the gradient $\langle\nabla\vec{\varphi}\rangle$ producing it:

$$\langle\vec{j}\rangle = -\Lambda\langle\nabla\vec{\varphi}\rangle, \quad \langle\nabla\vec{\varphi}\rangle = \frac{1}{V}\int_V \nabla\vec{\varphi}_i dV_i. \tag{1}$$

For local regions occupied by the i-th component the following relations are valid:

$$\vec{j}_i = -\Lambda_i\nabla\vec{\varphi}_i, \quad \operatorname{div}\vec{j}_i = 0. \tag{2}$$

Methods of calculating the conductivity of two-component heterogeneous systems as a function of the conductivities of the components Λ_i and their volume concentrations have been developed in adequate detail [1, 2].

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